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A SIMPLE THEORY OF PLASTICITY

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Abstract—An internal-variable model of rate-independent plastic behavior, based on loading–unloading irreversibility, is proposed. The model is compatible with thermodynamics and assumes no yield or loading function, stability postulate or specific nature of the internal variables. It is shown that current theories of plasticity are restricted forms of the proposed theory.

1. INTRODUCTION

The purpose of this note is to expose a theory of plasticity which is an outgrowth of some recent work of mine[1, 2] on the thermodynamics of materials in which irreversible processes are characterized by internal variables. The bulk of the literature on such materials[3–8] deals with internal variables that are governed by local rate equations of the form

$$\dot{q}_\alpha = f_\alpha(\text{state}), \qquad \alpha = 1, \ldots, n, \tag{1}$$

where n is the number of internal variables, and the state is given locally, for example, by the temperature θ, the Cauchy–Green deformation tensor $\mathbf{C}$, and the n-vector $\mathbf{q}$ whose components are the internal variables q_α (which may be scalars or components of tensors that are unchanged by rigid-body motion). Clearly the solutions of equation (1) are not invariant under time reversal. This kind of irreversibility, typical of rate processes, characterizes viscoelastic and viscoplastic materials, the difference between these two types of material behavior residing in the structure of the functions f_α.

Rate-independent plastic flow, on the other hand, is characterized by a different kind of irreversibility. It may be called a *one-way* process: it occurs when the external process (mechanical work and heating) goes in some direction (called loading) but not when it is reversed (unloading). If the state of external process is defined locally by $\mathbf{C}$ and θ, then we may assume the existence of a tensor $\mathbf{A}$, a scalar a, and an n-vector $\mathbf{r}$, all functions of the state variables $\mathbf{C}$, θ and $\mathbf{q}$, such that the loading rate ϕ is given by

$$\phi = \operatorname{tr}(\mathbf{A}\dot{\mathbf{C}}) + a\dot{\theta} \tag{2}$$

and the internal variables are governed by

$$\dot{\mathbf{q}} = \mathbf{r}\langle\phi\rangle \tag{3}$$

where $\langle u \rangle = uH(u)$, $H(u)$ being the Heaviside step function.

Alternatively we may take as external variables the temperature θ and the Piola–Kirchhoff stress tensor $\mathbf{P}$, and assume the existence of $\mathbf{B}$, b and $\mathbf{r}$, respectively tensor, scalar and n-vector functions of $\mathbf{P}$, θ and $\mathbf{q}$ such that

$$\phi = \mathrm{tr}(\mathbf{B}\dot{\mathbf{P}}) + b\dot{\theta} \tag{4}$$

and $\mathbf{q}$ is governed in form by equation (3) as before.

The proposed model of rate-independent plasticity is embodied in either equations (2) and (3) or (3) and (4). It does not require (but does not preclude) the existence of yield or loading surfaces, or of stability postulates, and it leaves the number and nature of the internal variables unspecified. The relationship between these traditional concepts and the present model will be explored in section 3. Since the model is one of irreversible behavior, it must first be examined for compatibility with the second law of thermodynamics.

2. COMPATIBILITY WITH THE SECOND LAW

The second law will be expressed under the guise of the material local form of the Clausius–Duhem inequality[9]. Since the functions characterizing plastic flow, as defined in the preceding section, are independent of temperature gradient we may limit the inequality to its form for vanishing temperature gradient. The existence of entropy and Helmholtz free energy per unit mass, denoted respectively by η and ψ, as functions of $\mathbf{C}$, θ and $\mathbf{q}$ will be assumed on the grounds that at any point in state space there are neighboring points that can be reached by unloading, that is, a process in which only $\mathbf{C}$ and θ vary; hence any state is one of constrained equilibrium.

With ρ_0 denoting mass density in the reference configuration and γ irreversible entropy production per unit mass, we have

$$-\rho_0(\dot{\psi} + \eta\dot{\theta}) + \tfrac{1}{2}\,\mathrm{tr}(\mathbf{P}\dot{\mathbf{C}}) = \rho_0\theta\gamma \geq 0. \tag{5}$$

With ψ a C' function, this becomes, upon application of the chain rule and substitution of equations (2) and (3),

$$-\rho_0\left[\frac{\partial\psi}{\partial\theta} + \eta + H(\phi)r_\alpha\frac{\partial\psi}{\partial q_\alpha}\alpha\right]\dot{\theta} + \mathrm{tr}\left\{\left[\tfrac{1}{2}\mathbf{P} - \rho_0\frac{\partial\psi}{\partial\mathbf{C}} - H(\phi)r_\alpha\rho_0\frac{\partial\psi}{\partial q_\alpha}\mathbf{A}\right]\dot{\mathbf{C}}\right\} = \rho_0\theta\gamma \geq 0. \tag{6}$$

The inequality is satisfied for all ϕ if and only if†

$$\begin{aligned} \eta &= -\frac{\partial\psi}{\partial\theta} - \lambda\frac{\partial\psi}{\partial q_\alpha}r_\alpha a, \\ \mathbf{P} &= 2\rho_0\left(\frac{\partial\psi}{\partial\mathbf{C}} + \lambda\frac{\partial\psi}{\partial q_\alpha}r_\alpha\mathbf{A}\right), \end{aligned} \tag{7}$$

provided

$$\frac{\partial\psi}{\partial q_\alpha}r_\alpha \leq 0 \tag{8}$$

and

$$0 \leq \lambda \leq 1, \tag{9}$$

† See Appendix for necessity proof.

since the inequality (6) then reduces to

$$\theta\gamma = \lambda \frac{\partial \psi}{\partial q_\alpha} r_\alpha \phi \geq 0, \qquad (\phi \leq 0) \tag{10a}$$

$$= -(1-\lambda) \frac{\partial \psi}{\partial q_\alpha} r_\alpha \phi \geq 0 \qquad (\phi > 0). \tag{10b}$$

It is significant that, unlike the models studied by Coleman and his collaborators[6, 9–11], for the present model the Clausius-Duhem inequality is not sufficient to determine constitutive equations for stress and entropy. To eliminate the indeterminacy presented by the parameter λ we must additionally argue that the unloading process is quasi-reversible, that is, $\gamma = 0$ when $\phi < 0$. Then, by (10a), we have $\lambda = 0$, and equations (7) may be replaced by the classical relations

$$\eta = -\frac{\partial \psi}{\partial \theta}, \qquad \mathbf{P} = 2\rho_0 \frac{\partial \psi}{\partial \mathbf{C}} = \rho_0 \frac{\partial \psi}{\partial \mathbf{E}}, \tag{11}$$

where $\mathbf{E} \equiv \frac{1}{2}(\mathbf{C} - \mathbf{I})$, while the irreversible entropy production is given by

$$\theta\gamma = -\frac{\partial \psi}{\partial q_\alpha} r_\alpha \langle \phi \rangle, \tag{12}$$

restricted to be non-negative by the inequality (8).

Analogous considerations apply to the complementary model given by equations (3) and (4), with $\mathbf{P}$, θ and $\mathbf{q}$ as state variables. The complementary free energy per unit mass, χ, may be defined by the Legendre transformation

$$\chi = \frac{1}{\rho_0} \operatorname{tr}(\mathbf{PE}) - \psi. \tag{13}$$

We then have

$$\mathbf{E} = \rho_0 \frac{\partial \chi}{\partial \mathbf{P}}, \qquad \eta = \frac{\partial \chi}{\partial \theta}, \tag{14}$$

and

$$\theta\gamma = \frac{\partial \chi}{\partial q_\alpha} r_\alpha \langle \phi \rangle \geq 0 \tag{15}$$

3. RELATIONSHIP TO OTHER THEORIES OF PLASTICITY

Among the salient *ad hoc* assumptions which are features of current theories of plasticity are the following:

(1) The existence of a *yield criterion*, given by a yield function, say $F(\mathbf{P}, \theta, \mathbf{q})$, such that no plastic flow occurs when $F < 0$, is assumed. In the context of the present model a yield function may be incorporated into the structure of the r_α, i.e. $\mathbf{r} = \mathbf{0}$ for $F < 0$.

(2) The existence of loading surfaces, given by $G(\mathbf{P}, \theta, \mathbf{q}) = 0$ such that the loading rate is given by

$$\phi = \mathrm{tr}\left(\frac{\partial G}{\partial \mathbf{P}}\dot{\mathbf{P}}\right) + \frac{\partial G}{\partial \theta}\dot{\theta}, \tag{16}$$

in other words, the existence of a function G, such that

$$\mathbf{B} = \frac{\partial G}{\partial \mathbf{P}}, \qquad b = \frac{\partial G}{\partial \theta} \tag{17}$$

is assumed. G may or may not coincide with F. The former view is the classical one, as formulated, for example, by Green and Naghdi[12], while the latter view was proposed originally by Melan[13] and more recently by Eisenberg and Phillips[14].

(3) The internal-variable vector is usually assumed to consist of the plastic strain tensor $\mathbf{E}^p$ and the hardening parameter κ[12], although, in place of the former, such other symmetric second-rank tensors as the contracted tearing curvature tensor of Kondo[15] or the dislocation loop density of Kröner[16] may represent more closely the internal processes. $\mathbf{E}^p$ arises naturally if the strain–stress equation (14a) has the property

$$\frac{\partial^2 \mathbf{E}}{\partial \mathbf{P}\,\partial \mathbf{q}} = \mathbf{0}, \qquad \frac{\partial^2 \mathbf{E}}{\partial \theta\,\partial \mathbf{q}} = \mathbf{0}, \tag{18}$$

since then

$$\mathbf{E} = \mathbf{E}^e(\mathbf{P}, \theta) + \mathbf{E}^p(\mathbf{q}) \tag{19}$$

(Equation (18) means that the thermoelastic moduli in terms of $\mathbf{P}$ and $\mathbf{E}$ are independent of plastic deformation.) If $n \geq 6$, then the components of $\mathbf{E}^p$ may be used as six of the internal variables (otherwise they will be subject to constraints).

Since by virtue of equation (14)

$$\frac{\partial E_{ij}}{\partial P_{kl}} = \frac{\partial E_{kl}}{\partial P_{ij}}, \tag{20}$$

equation (19) implies

$$\frac{\partial E_{ij}{}^e}{\partial P_{kl}} = \frac{\partial E_{kl}{}^e}{\partial P_{ij}}, \tag{21}$$

necessitating the existence of a function $\chi^e(\mathbf{P}, \theta)$ such that

$$\mathbf{E}^e = \rho_0 \frac{\partial \chi^e}{\partial \mathbf{P}} \tag{22}$$

(hence both E^e and E^p are symmetric), and χ is given by

$$\chi = \chi^e + \frac{1}{\rho_0}\mathrm{tr}(\mathbf{P}\mathbf{E}^p) + \chi^p(\theta, \mathbf{q}). \tag{23}$$

If furthermore the specific heat (at constant $\mathbf{P}$ and $\mathbf{q}$), given by

$$C = \theta\frac{\partial^2 \chi}{\partial \theta^2}, \tag{24}$$

is also assumed independent of $\mathbf{q}$, then χ^p is at most linear in θ, and χ is given by

$$\chi = \chi^e + \frac{1}{\rho_0}\,\mathrm{tr}(\mathbf{P}\mathbf{E}^p) + \theta\eta^p(\mathbf{q}) - \varepsilon^p(\mathbf{q}), \tag{25}$$

where ε^p is the additional internal energy (strain energy) and η^p is the configurational entropy due to plastic deformation[17]. Since the Clausius-Duhem inequality (15), with χ given by (25), reduces to

$$\mathrm{tr}(\mathbf{P}\dot{\mathbf{E}}^p) + \rho_0(\theta\dot{\eta}^p - \dot{\varepsilon}^p) \geq 0 \tag{26}$$

it is clear that such an entropy (increasing with plastic deformation) must exist if the Bauschinger effect (entailing the possibility that $\mathrm{tr}\,\mathbf{P}\dot{\mathbf{E}}^p < 0$) is to be compatible with the second law of thermodynamics.

With $\mathbf{q}$ given by $\{\mathbf{E}^p, \kappa\}$, $\mathbf{r}$ is given by, say, $\{\mathbf{M}, m\}$. Two definitions of κ are common: the first as the plastic work, so that

$$\dot{\kappa} = \mathrm{tr}(\mathbf{P}\dot{\mathbf{E}}^p), \tag{27}$$

implying

$$m = \mathrm{tr}(\mathbf{P}\mathbf{M}), \tag{28}$$

and the second as given by

$$\dot{\kappa} = [\mathrm{tr}(\dot{\mathbf{E}}^p\dot{\mathbf{E}}^p)]^{1/2}, \tag{29}$$

so that

$$m = (\mathrm{tr}\,\mathbf{M}^2)^{1/2}. \tag{30}$$

Note that Green & Naghdi[12] assume $\dot{\kappa}$ linear (and not merely homogeneous of the first degree) in $\dot{\mathbf{E}}^p$, precluding the definition (29).

(4) Drucker's postulate of material stability[18] takes, in terms of $\mathbf{P}$ and $\mathbf{E}$, the form

$$\mathrm{tr}\,\dot{\mathbf{P}}\dot{\mathbf{E}}^p \geq 0. \tag{31}$$

In conjunction with the concepts of yield and loading surfaces, this postulate has been used to prove the convexity of these surfaces and the normality of $\dot{\mathbf{E}}^p$ to the loading surface. However, the postulate itself is independent of the existence of any such surfaces and even of the plastic strain $\mathbf{E}^p$, since a plastic strain rate $\dot{\mathbf{E}}^p$ may always be defined as

$$\begin{aligned}\dot{\mathbf{E}}^p &= \frac{\partial \mathbf{E}}{\partial q_\alpha}\dot{q}_\alpha \\ &= \mathbf{M}\langle\phi\rangle,\end{aligned} \tag{32}$$

where

$$\mathbf{M} = \frac{\partial \mathbf{E}}{\partial q_\alpha} r_\alpha. \tag{33}$$

Taking (31) as valid isothermally, we have

$$\mathrm{tr}(\mathbf{M}\dot{\mathbf{P}})\mathrm{tr}(\mathbf{B}\dot{\mathbf{P}}) > 0, \tag{34}$$

which requires, in order to be valid for all $\dot{\mathbf{P}}$,

$$\mathbf{M} = \mu\mathbf{B} \tag{35}$$

where $\mu > 0$. The "normality rule" follows if $\mathbf{B}$ is given by (17a).

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APPENDIX—PROOF OF NECESSITY OF EQUATIONS (7)

Let $\left\{\frac{1}{2\rho_0}\mathbf{P} - \frac{\partial\psi}{\partial\mathbf{C}}, -\left(\eta + \frac{\partial\psi}{\partial\theta}\right)\right\}$, $\{\mathbf{A}, a\}$, and $\{\dot{\mathbf{C}}, \dot{\theta}\}$ be denoted by the 7-vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, respectively, and $r_\alpha\, \partial\psi/\partial q_\alpha$ by $-k$. With this notation the Clausius–Duhem inequality may be written as

$$\mathbf{u}\cdot\mathbf{w} + k\langle\mathbf{v}\cdot\mathbf{w}\rangle \geq 0, \tag{A1}$$

and equations (7) are equivalent to the assertion that

$$\mathbf{u} = -\lambda k\mathbf{v}, \tag{A2}$$

that is, $\mathbf{u}$ is parallel to $\mathbf{v}$. To prove that (A1) implies (A2) we assume the contrary of the latter, that is, we suppose that $\mathbf{u}$ also has a component normal to $\mathbf{v}$:

$$\begin{aligned} \mathbf{u} &= -\lambda k \mathbf{v} + \mathbf{z}, \\ \mathbf{v} \cdot \mathbf{z} &= 0. \end{aligned} \tag{A3}$$

Now suppose $\mathbf{w} = c\mathbf{z}$, where c may be any real number. (A1) and (A3) together imply

$$c\mathbf{z} \cdot \mathbf{z} > 0. \tag{A4}$$

For (A4) to hold for all real c, $\mathbf{z}$ must vanish, that is, (A2) is necessary.

Абстракт — На основе процесса необратимости нагрузки и разгрузки, предполагается модель пластического поведения, независящого от скорости, которая ползуется внутренными переменными. Модель совместима с термодинамикой и не предполагает функции течения или нагрузки, постулата устойчивости или специфической природы внутренных переменных. Указано, что текущие теории пластичности оказываются ограниченными формами предлагаемой теории.